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g^*-Angle Between Two Subspaces in The Space of p-Summable Sequences

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Abstract. We define a new semi-inner product g^* on l^p spaces for $1 \le p < \infty$ equipped with a 2-norm. Using g^*, we study the g^*-angle between two vectors on l^p . We also develop the notion of the g^*-angle between a 1-dimensional subspace and an k-dimensional subspace for $k \ge 1$ in the 2-normed space.

INTRODUCTION

Let $(V, \|\cdot\|)$ be a real normed space. The functional $g: V \times V \to \mathbb{R}$ defined by

$$g(u,v) := \frac{1}{2}\|u\|[\tau_+(u,v) + \tau_-(u,v)]$$

where

$$\tau_\pm(u,v) := \lim_{h\to 0^\pm} \frac{\|u+hv\| - \|u\|}{h}.$$

We know that $g(u,v)$ satisfies the following basic properties [5]:

1) $g(u,u) = \|u\|^2$ for any $u \in V$;
2) $g(\alpha u, \beta v) = \alpha\beta g(u,v)$ for any $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$;
3) $g(u, u+v) = \|u\|^2 + g(u,v)$ for any $u, v \in V$;
4) $|g(u,v)| \le \|u\|\|v\|$ for any $u, v \in V$.

In general, $g(u,v)$ is not linear in v. If, in addition to the above three properties $g(u,v)$ is linear in v, then g is called a *semi-inner product* on V. For instance, the function

$$g(u,v) = \|u\|_p^{2-p} \sum_j |u_j|^{p-1} \mathrm{sgn}(u_j) v_j, \quad u := (u_j), v := (v_j) \in l^p \qquad (1)$$

is a semi-inner product on $(l^p, \|\cdot\|_p)$ for $1 \le p < \infty$ [5].

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Many researchers have studied g-angle between two vectors in V using the semi-inner product g, see, for instance [1,9,10,11]. In 2018, Nur *et al*. [14] develop the notion of the g-angle between two subspaces of V. If $S = \text{span}\{s\}$ is a 1-dimensional subspace and $T = \text{span}\{t_1, \dots, t_k\}$ is an k-dimensional subspace of V with $k \geq 1$, then the g-angle between subspaces S and T is defined by $A_g(S,T)$ with $\cos^2 A_g(S,T) = \frac{(g(s_T,s))^2}{\|s\|^2\|s_T\|^2}$. In the formula, s_T is the g-orthogonal projection of s on T. Moreover, Nur *et al*. show that the value of $\cos A_g(S,T)$ is equal to the ratio between the 'length' of the g-orthogonal projection of s on T and the 'length' of s $\left(\cos^2 A_g(S,T) = \frac{\|s_T\|^2}{\|s\|^2}\right)$. Recently, Nur and Gunawan [13] define the angle between 2-dimensional subspaces by using a 2-norm.

In general, a 2-norm on a vector space V is a mapping $\|\cdot,\cdot\|: V \times V \to \mathbb{R}$ which satisfies the following four conditions:

(1) $\|u, v\| = 0$ if and only if u, v are linearly dependent;
(2) $\|u, v\| = \|v, u\|$ for any $u, v \in V$;
(3) $\|\alpha u, v\| = |\alpha|\|u, v\|$, for any $u, v \in V$ and for any $\alpha \in \mathbb{R}$;
(4) $\|u, v + w\| \leq \|u, v\| + \|u, w\|$, for any $u, v, w \in V$.

Next, the part $(V, \|\cdot,\cdot\|)$ is called a *2-normed space.*

Geometrically, 2-norm $\|u, v\|$ may be interpreted as the area of the 2-dimensional parallelepiped spanned by u, v. The notion of 2-normed space was first developed in the mid 1960's by Gähler [4]. Recent results can be found, for instance, in [2,3,7,8,12]. On the space l^p for $1 \leq p < \infty$, the following the 2-norm was defined by Gunawan in [6],

$$\|u, v\|_p = \left[\frac{1}{2}\sum_j \sum_k \left(\text{abs}\begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix}\right)^p\right]^{\frac{1}{p}} \tag{2}$$

where $u \coloneqq (u_j), v \coloneqq (v_j) \in l^p$.

In this article, we will define a semi-inner product g^* on $(l^p, \|\cdot,\cdot\|_p)$ with $1 \leq p < \infty$. Next, using a semi-inner product g^*, we can introduce the g^*-angle between two vectors. Moreover, we can study g^*-angle between a 1-dimensional subspace and an k-dimensional subspace for $k \geq 1$ in the 2-normed space $(l^p, \|\cdot,\cdot\|_p)$.

MAIN RESULTS

g^*-Angle Between Two Vectors

In this subsection, we will discuss g^*-angle between two vectors in the 2-normed space $(l^p, \|\cdot,\cdot\|_p)$. Let $\{a_1, a_2\}$ be a linearly independent set on l^p. Firstly, we define the following function.

$$\|u\|_p^* = \left[\|u, a_1\|_p^p + \|u, a_2\|_p^p\right]^{\frac{1}{p}}, \tag{3}$$

for every $u \in l^p$. Next, we have the following proposition.

Proposition 2.1. *[6] The mapping $\|\cdot\|_p^*$ defines a norm on l^p.*

Using the norm $\|\cdot\|_p^*$ with $a_1 = (a_{1j})$ and $a_2 = (a_{2j})$, we define a mapping $g^*(\cdot,\cdot)$ on the 2-normed space $(l^p, \|\cdot,\cdot\|_p)$ with $1 \leq p < \infty$ by

$$g^*(u, v) = \frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_j \sum_k \left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}\right)\begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix} \tag{4}$$

for every $u = (u_j), v = (v_j) \in l^p$.

Then we have the following result.

Theorem 2.2. *The mapping $g^*(u,v)$ in (4) defines a semi-inner product on $\left(l^p, \|\cdot,\cdot\|_p\right)$.*

Proof. We will show that $g^*(u,v)$ satisfies the properties of g and linear in terms of v.

1. Observe that

$$g^*(u,u) = \frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}$$
$$= \left(\|u\|_p^*\right)^{2-p}\left[\|u,a_1\|_p^p + \|u,a_2\|_p^p\right]$$
$$= \left(\|u\|_p^*\right)^2.$$

2. Observe that

$$g^*(\alpha u,\beta v) = \frac{\left(\|\alpha u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} \alpha u_j & \alpha u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} \alpha u_j & \alpha u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)\begin{vmatrix} \beta v_j & \beta v_k \\ a_{ij} & a_{ik}\end{vmatrix}$$
$$= \alpha\beta\frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)\begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik}\end{vmatrix}$$
$$= \alpha\beta g^*(u,v).$$

3. Using properties of determinants, we obtain

$$g^*(u,u+v) = \frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)\begin{vmatrix} u_j+v_j & u_k+v_k \\ a_{ij} & a_{ik}\end{vmatrix}$$
$$= \frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}$$
$$+ \frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)\begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik}\end{vmatrix}$$
$$= \left(\|u\|_p^*\right)^2 + g^*(u,v).$$

4. Observe that

$$|g^*(u,v)| = \left|\frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)\begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik}\end{vmatrix}\right|$$
$$\leq \frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{abs}\begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)$$
$$\leq \left(\|u\|_p^*\right)^{2-p}\left[\frac{1}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p}\right]^{\frac{p-1}{p}}\left[\frac{1}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p}\right]^{\frac{1}{p}}$$
$$= \|u\|_p^*\|v\|_p^*.$$

5. Observe that

$$g^*(u,v+v') = \frac{\left(\|u\|_p^*\right)^{2-p}}{2}\sum_{i=1}^{2}\sum_{j}\sum_{k}\left(\text{abs}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)^{p-1}\left(\text{sgn}\begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik}\end{vmatrix}\right)\begin{vmatrix} v_j+v_j{}' & v_k+v' \\ a_{ij} & a_{ik}\end{vmatrix}$$

$$= \frac{\left(\|u\|_p^*\right)^{2-p}}{2} \sum_{i=1}^{2} \sum_{j} \sum_{k} \left(\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}\right)^{p-1} \left(\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}\right) \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix}$$
$$+ \frac{\left(\|u\|_p^*\right)^{2-p}}{2} \sum_{i=1}^{2} \sum_{j} \sum_{k} \left(\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}\right)^{p-1} \left(\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}\right) \begin{vmatrix} v_j' & v_k' \\ a_{ij} & a_{ik} \end{vmatrix}$$
$$= g^*(u,v) + g^*(u,v').$$

Therefore, $g^*(u,v)$ defines the semi-inner product on $\left(l^p, \|\cdot,\cdot\|_p\right)$. ■

Remark 2.3. The functional $g^*(\cdot,\cdot)$ does not satisfy commutative property. For example, consider l^1 with $a_1 = (1,0,0,\dots)$, $a_2 = (0,1,0,\dots)$ and $\|\cdot\|_1^*$ in Proposition 2.1. Take $u = (1,1,3,0,\dots)$ and $v = (2,1,-3,0,\dots)$. Clearly $\|u\|_1^* = 8$ and $\|v\|_1^* = 9$. Thus $g^*(v,u) = -45 \neq g^*(u,v) = -16$.

Next, using (4) and definition of g-angle in [14], we have the g^*-angle between vectors u and v on $\left(l^p, \|\cdot,\cdot\|_p\right)$ as follows.

$$A_g^*(u,v) = \arccos \frac{g^*(v,u)}{\|u\|_p^* \|v\|_p^*}. \tag{5}$$

Note that $A_g^*(u,v) = \frac{1}{2}\pi$ if and only if $g^*(v,u) = 0$ or $v \perp_g u$.

Example 2.4. Let $(l^1, \|\cdot,\cdot\|_1)$ be 2 normed space with $a_1 = (1,0,0,\dots)$, and $a_2 = (0,1,0,\dots)$. If $u = (1,2,1,0,\dots)$ and $v = (2,1,3,0,\dots)$ then $\|u, a_1\|_1 = 3$, $\|u, a_2\|_1 = 2$, $\|y, a_1\|_1 = 4$ and $\|v, a_2\|_1 = 5$. Clearly $\|u\|_1^* = 5$ and $\|v\|_1^* = 9$. Moreover, using g^* in Theorem 2.2 for $p = 1$, we obtain

$$g^*(v,u) = \frac{\|v\|_p^*}{2} \sum_{i=1}^{2} \sum_{j} \sum_{k} \left(\text{sgn} \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix}\right) \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}$$
$$= \frac{5}{2}(10) = 25.$$

Hence, $A_g^*(u,v) = \arccos\left(\frac{5}{9}\right)$.

g^*-Angle Between a 1-Dimensional Subspace and an k-Dimensional Subspace

In this part, we will define the $\boldsymbol{g}^*$-angle between a **1**-dimensional subspace and an $\boldsymbol{k}$-dimensional subspace in $\left(\boldsymbol{l^p}, \|\cdot,\cdot\|_{\boldsymbol{p}}\right)$. We will apply the definition of $\boldsymbol{g}$-angle in [14] with a new norm $\|\cdot\|_{\boldsymbol{p}}^*$ that is obtained from the **2**-norm. Using $\boldsymbol{g}^*(\cdot,\cdot)$ in (4) , we have the Gram determinant $\boldsymbol{\Gamma_{g^*}(t_1, \dots, t_n) = \det\left[g^*(t_i, t_j)\right]}$, where $\boldsymbol{g^*(t_i, t_j)}$ is the $\boldsymbol{j}$-th element of the $\boldsymbol{i}$-th row. Moreover, we have the $\boldsymbol{g}^*$-orthogonal projection of $\boldsymbol{s}$ on $\boldsymbol{T}$ as follows.

Definition 2.5 Let s be a vector of l^p and $T = \text{span}(t_1, \dots, t_n)$ be subspace of l^p with $\Gamma_{g^*}(t_1, \dots, t_n) \neq 0$. The g^*-*orthogonal projection* of s on T (s_{T^*}) is defined by

$$s_{T^*} = -\frac{1}{\Gamma_{g^*}(t_1, \dots, t_n)} \begin{vmatrix} 0 & t_1 & \cdots & t_n \\ g^*(t_1, s) & g^*(t_1, t_1) & \cdots & g^*(t_1, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ g^*(t_n, s) & g^*(t_n, t_1) & \cdots & g^*(t_n, t_n) \end{vmatrix}.$$

By using s_{T^*} in Definition 2.8, we have g^*-*orthogonal complement* $s_{T^*}^{\perp}$ as follows.

$$\boldsymbol{s}_{T^*}^{\perp} = \boldsymbol{s} - \boldsymbol{s}_{T^*} = \frac{\mathbf{1}}{\boldsymbol{\Gamma}_{g^*}(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n)} \begin{vmatrix} \boldsymbol{s} & \boldsymbol{t}_1 & \cdots & \boldsymbol{t}_n \\ \boldsymbol{g}^*(\boldsymbol{t}_1, \boldsymbol{s}) & \boldsymbol{g}^*(\boldsymbol{t}_1, \boldsymbol{t}_1) & \cdots & \boldsymbol{g}^*(\boldsymbol{t}_1, \boldsymbol{t}_n) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{g}^*(\boldsymbol{t}_n, \boldsymbol{s}) & \boldsymbol{g}^*(\boldsymbol{t}_n, \boldsymbol{t}_1) & \cdots & \boldsymbol{g}^*(\boldsymbol{t}_n, \boldsymbol{t}_n) \end{vmatrix}.$$

Next, we write the $\boldsymbol{g}^*$-angle between $\boldsymbol{S} = \mathbf{span}(\boldsymbol{s})$ and $\boldsymbol{T} = \mathbf{span}(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_k)$ for $\boldsymbol{k} \geq \mathbf{1}$ of $(\boldsymbol{l}^p, \|\cdot,\cdot\|_p)$ as follows.

Definition 2.6. Let $(l^p, \|\cdot,\cdot\|_p)$ be a 2-normed space for $1 \leq p < \infty$. If $S = \mathrm{span}(s)$ is a 1-dimensional subspace and $T = \mathrm{span}(t_1, \ldots, t_k)$ is an k-dimensional subspace of l^p with $\Gamma_{g^*}(t_1, \ldots, t_k) \neq 0$ dan $k \geq 1$ then g^*-angle between S and T denoted by $A_g^*(S,T)$ with

$$\cos^2 A_g^*(S,T) = \frac{\left(g^*(s_{T^*}, s)\right)^2}{\left(\|s_{T^*}\|_p^*\right)^2 \left(\|s\|_p^*\right)^2}, \tag{6}$$

where s_{T^*} is the g^*-ortogonal projection of s on T.

Next, by using the formula $s = s_{T^*} + s_{T^*}^{\perp}$, we have the value of $\cos^2 A_g^*(S,T)$ by

$$\cos^2 A_g^*(S,T) = \frac{\left(g^*\left(s_{T^*}, s_{T^*} + s_{T^*}^{\perp}\right)\right)^2}{\left(\|s_{T^*}\|_p^*\right)^2 \left(\|s\|_p^*\right)^2} = \frac{\left(g^*(s_{T^*}, s_{T^*}) + g^*\left(s_{T^*}, s_{T^*}^{\perp}\right)\right)^2}{\left(\|s_{T^*}\|_p^*\right)^2 \left(\|s\|_p^*\right)^2}$$
$$= \frac{\left(g^*(s_{T^*}, s_{T^*})\right)^2}{\left(\|s_{T^*}\|_p^*\right)^2 \left(\|s\|_p^*\right)^2} = \frac{\left(\|s_{T^*}\|_p^*\right)^2}{\left(\|s\|_p^*\right)^2}.$$

Example 2.7. Let $(l^1, \|\cdot,\cdot\|_p)$ be a 2-normed space with $a_1 = (1,0,0, \ldots)$, and $a_2 = (0,1,0, \ldots)$. If $S = \mathrm{span}(s)$ and $T = \mathrm{span}(t_1, t_2)$ with $s = (1,1,3,0, \ldots)$, $t_1 = (1,0,0,0, \ldots)$ and $t_2 = (0,1,0,0, \ldots)$ then $\|s\|_1^* = 8$, $\|t_1\|_1^* = 1$, and $\|t_2\|_1^* = 1$. Using the semi inner product-g in Theorem 2.2, we have

$$\Gamma_{g^*}(t_1, t_2) = \begin{vmatrix} g^*(t_1, t_1) & g^*(t_1, t_2) \\ g^*(t_2, t_1) & g^*(t_2, t_2) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

As a consequence, we obtain

$$s_{T^*} = -\frac{1}{\Gamma_{g^*}(t_1, t_2)} \begin{vmatrix} 0 & t_1 & t_2 \\ g^*(t_1, s) & g^*(t_1, t_1) & g^*(t_1, t_2) \\ g^*(t_2, s) & g^*(t_2, t_1) & g^*(t_2, t_2) \end{vmatrix}$$
$$= -\begin{vmatrix} 0 & t_1 & t_2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = t_1 + t_2 = (1,1,0, \ldots).$$

Next, we can compute $\|s_{T^*}\|_1^*$ by

$$\|s_{T^*}\|_1^* = \frac{1}{2} \sum_j \sum_k \left(\mathrm{abs} \begin{vmatrix} s_{T^*j} & s_{T^*k} \\ a_{1j} & a_{1k} \end{vmatrix}\right) + \frac{1}{2} \sum_j \sum_k \left(\mathrm{abs} \begin{vmatrix} s_{T^*j} & s_{T^*k} \\ a_{2j} & a_{2k} \end{vmatrix}\right)$$
$$= \frac{1}{2} \sum_j \left(\mathrm{abs} \begin{vmatrix} s_{T^*j} & 1 \\ a_{1j} & 1 \end{vmatrix}\right) + \left(\mathrm{abs} \begin{vmatrix} s_{T^*j} & 1 \\ a_{1j} & 0 \end{vmatrix}\right) + \frac{1}{2} \sum_j \left(\mathrm{abs} \begin{vmatrix} s_{T^*j} & 1 \\ a_{1j} & 0 \end{vmatrix}\right) + \left(\mathrm{abs} \begin{vmatrix} s_{T^*j} & 1 \\ a_{2j} & 1 \end{vmatrix}\right)$$
$$= 1 + 1 = 2.$$

Thus $\cos^2 A_g^*(S,T) = \frac{4}{64}$, so that $A_g^*(S,T) = \arccos\left(\frac{1}{4}\right)$.

CONCLUDING REMARKS

We can extend our result for L^p spaces. Let V be a measure space and f is measurable function. The function $f \in L^p(V)$ if $\int_V |f(v)|^p dv < \infty$. Next, the function $\|f\|_{L^p} = \left(\int_V |f(v)|^p dv\right)^{\frac{1}{p}}$ defines a norm on $L^p(V)$. Gunawan [6] defines a 2-norm $\|\cdot,\cdot\|_{L^p}$ on $L^p(X) \times L^p(X)$ by

$$\|\boldsymbol{f_1}, \boldsymbol{f_2}\|_{\boldsymbol{L^p}} = \left[\frac{\mathbf{1}}{\mathbf{2}}\int_V \int_V \mathbf{abs}\left(\begin{vmatrix} \boldsymbol{f_1(v_1)} & \boldsymbol{f_1(v_2)} \\ \boldsymbol{f_2(v_1)} & \boldsymbol{f_2(v_2)} \end{vmatrix}\right) \boldsymbol{dv_1 dv_2}\right]^{\frac{\mathbf{1}}{\boldsymbol{p}}} \qquad (\mathbf{7})$$

According to the 2-norm in (7), Ekariani et.al [3] defines a new norm on $L^p(V)$ by

$$\|f\|^*_{L^p} = \left[\|f, a_1\|^p_{L^p} + \|f, a_2\|^p_{L^p}\right]^{\frac{1}{p}},$$

where $\{a_1, a_2\}$ be a linearly independent set on $L^p(V)$. Using the norm $\|\cdot\|^*_{L^p}$, one may define $g^*(\cdot,\cdot)$ on $(L^p(V), \|\cdot,\cdot\|_{L^p})$ by

$$g^*(f_1, f_2) = \frac{(\|f_1\|^*_{L^p})^{2-p}}{2} \sum_{i=1}^{2} \int_X \int_X \left(\text{abs}\begin{vmatrix} f_1(v_1) & f_1(v_2) \\ a_i(v_1) & a_i(v_2) \end{vmatrix}\right)^{p-1} \left(\text{sgn}\begin{vmatrix} f_1(v_1) & f_1(v_2) \\ a_i(v_1) & a_i(v_2) \end{vmatrix}\right) \begin{vmatrix} f_2(v_1) & f_2(v_2) \\ a_i(v_1) & a_i(v_2) \end{vmatrix} \, dv_1 dv_2 \, .$$

and check that this mapping defines a semi inner product on $L^p(V)$. The analogues in (5), we have the g^*-angle two vectors on $L^p(V)$. Moreover, the analogues in (6), we have the g^*-angle in a 2-normed space $(L^p(V), \|\cdot,\cdot\|_{L^p})$**.**

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